Eur. Phys. J. B **22**, 65–69 (2001)

THE EUROPEAN PHYSICAL JOURNAL B

EDP Sciences

Controlling chaos by a modified straight-line stabilization method

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Received 10 January 2001

Abstract. By adjusting external control signal, rather than some available parameters of the system, we modify the straight-line stabilization method for stabilizing an unstable periodic orbit in a neighborhood of an unstable fixed point formulated by Ling Yang *et al.*, and derive a more simple analytical expression of the external control signal adjustment. Our technique solves the problem that the unstable fixed point is independent of the system parameters, for which the original straight-line stabilization method is not suitable. The method is valid for controlling dissipative chaos, Hamiltonian chaos and hyperchaos, and may be most useful for the systems in which it may be difficult to find an accessible system parameter in some cases. The method is robust under the presence of weak external noise.

PACS. 05.45.+a Nonlinear dynamics and nonlinear dynamical systems – 05.45.Gg Control of chaos, applications of chaos – 05.45.Pq Numerical simulations of chaotic models

The control of chaos is an interesting subject in nonlinear dynamics. A wide variety of methods have been proposed for controlling chaos in nonlinear dynamical systems since Ott, Grebogi and Yorke (OGY) gave a method to stabilize an unstable periodic orbit by a small perturbation [1]. However, a large number of works in the literatures so far have concentrated on dissipative systems [2–7]. For conservative systems, controlling chaos is more difficult because there are no chaotic attractors and the search for chaotic behavior involves large areas of the phase space. The initial conditions are special controlling parameters and have important roles in describing chaotic behavior. Except for the extension of the OGY method by Lai *et al.* [8], a general way is lack for controlling Hamiltonian chaos. Recently, Ling Yang *et al.* [9,10] proposed the straight-line stabilization method for controlling hyperchaos which is to guide an unstable orbit in a neighborhood of a "fully" unstable fixed point to go to the fixed point directly along the straight line connecting the orbit (at a given time) and the fixed point. The method does not require any previous knowledge of the system. In this paper, we modify the straight-line stabilization method by adjusting the external control signal, rather than some available parameters of the system, and derive a more simple expression of external control signal adjustment. Our technique solves the problem that the unstable fixed point is independent of the system parameters, for which the original straight-line stabilization method is not suitable. The method is valid for controlling dissipative chaos, Hamiltonian chaos and hyperchaos. In particular, it might be of application in those situations where find an accessible system parameter may be a difficult task, namely, the case of certain chemical, biological, or spatially nonhomogeneous systems.

We consider the following discrete time dynamical system

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{F}$ is a sufficiently smooth function of $\mathbf{x}$. We act the external control signal on the system in the following form

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n, \epsilon) = \mathbf{F}(\mathbf{x}_n) + \epsilon, \tag{2}$$

where $\epsilon \in \mathbb{R}^N$ is the external control signal.

Let $\mathbf{x}_*^0$ be the fixed point of the map with $\epsilon = \mathbf{0}$. As long as the determinant of the Jacobian matrix $\mathbf{J} = (\partial \mathbf{F}(\mathbf{x}_n)/\partial \mathbf{x}_n)_{\mathbf{x}_n = \mathbf{x}_*^0}$ is not equal to zero, the map (2) with small control signal ϵ has a fixed point in the neighborhood of $\mathbf{x}_*^0$. Denote this fixed point by $\mathbf{x}_*$ and expand it to the first order about $\epsilon = \mathbf{0}$, *i.e.*

$$\mathbf{x}_* - \mathbf{x}_*^0 = \mathbf{M}\epsilon, \tag{3}$$

where $\mathbf{M} = (\partial \mathbf{x}_*/\partial \epsilon)_{\epsilon=\mathbf{0}}$. Meanwhile, the fixed point $\mathbf{x}_*$ with small control signal ϵ satisfies the equation $\mathbf{x}_* = \mathbf{F}(\mathbf{x}_*) + \epsilon$. We expand $\mathbf{F}(\mathbf{x}_*)$ to the first order about $\mathbf{x}_* = \mathbf{x}_*^0$: $\mathbf{F}(\mathbf{x}_*) = \mathbf{F}(\mathbf{x}_*^0) + \mathbf{J}(\mathbf{x}_* - \mathbf{x}_*^0) = \mathbf{x}_*^0 + \mathbf{J}(\mathbf{x}_* - \mathbf{x}_*^0)$, and get

$$\mathbf{x}_* - \mathbf{x}_*^0 = -(\mathbf{J} - \mathbf{I})^{-1}\epsilon, \tag{4}$$

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where $\mathbf{I}$ is the identity matrix. Comparing equations (3) and (4), we have

$$\mathbf{M} = -(\mathbf{J} - \mathbf{I})^{-1}. \tag{5}$$

It is noticed that the map (2) avoids the case of $\mathbf{M} = \mathbf{0}$, in other words, our technique solves the problem which the unstable fixed point is independent of the system parameters. However, it is not always the case in equation (1) of reference [9]. This means the original straight-line stabilization method has limitations in the sense.

The straight-line stabilization method [9,10] requires

$$\mathbf{x}_{n+1} - \mathbf{x}_*^0 = k(\mathbf{x}_n - \mathbf{x}_*^0), \tag{6}$$

where k is a constant and $-1 < k < 1$. We expand $\mathbf{x}_{n+1}$ to the first order about $\mathbf{x}_n = \mathbf{x}_*$, *i.e.*

$$\mathbf{x}_{n+1} - \mathbf{x}_* = \tilde{\mathbf{J}}(\mathbf{x}_n - \mathbf{x}_*), \tag{7}$$

where $\tilde{\mathbf{J}} = (\partial \mathbf{F}(\mathbf{x}_n, \epsilon)/\partial \mathbf{x}_n)_{\mathbf{x}_n = \mathbf{x}_*}$ is the Jacobian matrix of the map with small ϵ at $\mathbf{x}_n = \mathbf{x}_*$. The matrix $\tilde{\mathbf{J}}$ can be approximated by matrix $\mathbf{J}$. Hence, for $\epsilon \to \mathbf{0}$,

$$\mathbf{x}_{n+1} - \mathbf{x}_* = \mathbf{J}(\mathbf{x}_n - \mathbf{x}_*). \tag{8}$$

Using equations (3, 5, 6, 8) to eliminate $\mathbf{x}_{n+1}$ and $\mathbf{x}_*$, we obtain

$$\epsilon = \epsilon_n = (k\mathbf{I} - \mathbf{J})(\mathbf{x}_n - \mathbf{x}_*^0). \tag{9}$$

Here ϵ has been replaced by ϵ_n to indicate that the control signal adjustment is in the nth iteration of the map, and $\mathbf{x}_n$ is not necessarily close to $\mathbf{x}_*^0$. The analytical expression of the control signal adjustment depends only on $\mathbf{J}$. This is the major advantage of our method, which presents a more convenient approach for controlling chaos. Furthermore, in the straight-line stabilization method the unstable orbit is forced to go directly towards the fixed point itself, and not *via* the stable manifold. This implies that the method presents a possible solution to the problem of long chaotic transient [1,6]. In dissipative systems, this may not be as serious a problem because after a relatively short chaotic transient, the trajectory will move back to the desired controlling region. In Hamiltonian systems, however, the trajectory may experience an extremely long transient before it come close to the controlling region.

The above analysis can be extended to periodic orbits with period greater than one. The most direct way is to take the Tth iteration of the map, where T denotes the period of the orbit to be stabilized. For the T times iterated map, any point on the periodic orbit is a fixed point, and we can then apply the above discussion where $\mathbf{F}(\mathbf{x}_n)$ may be replaced by $\mathbf{F}^T(\mathbf{x}_n)$ in equation (2). We rewrite equations (2) and (9) as

$$\mathbf{x}_{n+1} = \mathbf{F}^T(\mathbf{x}_n, \epsilon) = \mathbf{F}^T(\mathbf{x}_n) + \epsilon, \tag{10}$$

$$\epsilon = \epsilon_n = (k\mathbf{I} - \mathbf{J}^T)(\mathbf{x}_n - \mathbf{x}_*^0). \tag{11}$$

Here $\mathbf{x}_*^0$ is one of the fixed points of the unstable period-T orbit, and $\mathbf{J}^T = (\partial \mathbf{F}^T(\mathbf{x}_n)/\partial \mathbf{x}_n)_{\mathbf{x}_n = \mathbf{x}_*^0}$.

For an unstable period-T orbit to be stabilized, it is worthwhile to indicate the analytical expression of the control signal ϵ if the analytical expression of $\mathbf{F}(\mathbf{x}_n)$ is known and that of $\mathbf{F}^T(\mathbf{x}_n)$ is difficult to obtain. In numerical calculation, we can write the control signal in the form

$$\begin{aligned}\mathbf{x}_{n+1} &= \mathbf{F}(\mathbf{x}_n, \epsilon) \\ &= \mathbf{F}(\mathbf{x}_n) + \epsilon_{n+1-T} \sum_{m=1}^{\infty} \delta(n + 1 - mT),\end{aligned} \tag{12}$$

where $\epsilon_{n+1-T} = (k\mathbf{I} - \mathbf{J}^T)(\mathbf{x}_{n+1-T} - \mathbf{x}_*^0)$, and $\mathbf{J}^T = \frac{\partial \mathbf{F}(\mathbf{x}_*^{T-1})}{\partial \mathbf{x}_{n+T-1}} \frac{\partial \mathbf{F}(\mathbf{x}_*^{T-2})}{\partial \mathbf{x}_{n+T-2}} \cdots \frac{\partial \mathbf{F}(\mathbf{x}_*^0)}{\partial \mathbf{x}_n}$ for this consequence $(\mathbf{x}_*^0, \mathbf{x}_*^1, \cdots, \mathbf{x}_*^{T-1})$ generated by successive iteration of the initial condition $\mathbf{x}_*^0$. In fact, equation (12) is equivalent to equation (10).

Although the discrete time dynamical system is discussed in the above, this method can be generalized to the continuous time dynamical systems. We may write the equations describing the system in iteration form by employing a fourth-order Runge-Kutta method

$$\delta \mathbf{x}_{n+1} = J_n \delta \mathbf{x}_n. \tag{13}$$

Here the step length has been carefully chosen in order to avoid spurious behavior. The behavior of the system will be simplified by constructing a proper Poincaré section. If the position of the system piercing the section from the same side along the trajectory every interval T times is the same point, the point is called the fixed point of the period-T orbit. Using the above method, the unstable period-T orbit can be also stabilized.

The method can be used in practical problems, for which any previous analytical knowledge of the system dynamics is usually not available, because the elements of the Jacobian matrix are experimentally accessible. The region where the stabilization method is valid can also be mathematically estimated [10].

In the following, we first demonstrate the original straight-line stabilization method and the modified one with the Henon map in which both methods can be applied, to compare their behaviors. The Henon map [11] is in the form

$$\begin{aligned}x_{n+1} &= 1 - a x_n^2 + y_n, \\ y_{n+1} &= b x_n.\end{aligned} \tag{14}$$

In the original straight-line stabilization method, the parameters a and b are the control parameters which can be written in the form

$$a = a_0 + \delta a, \qquad b = b_0 + \delta b. \tag{15}$$

Here a_0, b_0 are nominal parameter values, and δa, δb are small perturbations of the parameters. In this section, we consider $a_0 = 1.4, b_0 = 0.3$. When $\epsilon = (\delta a, \delta b) = (0, 0)$, an unstable fixed point is at $(x_*^0, y_*^0) = (0.63135, 0.18941)$. It is easy to get that $\mathbf{J} = \begin{pmatrix} -2a_0 x_*^0 & 1 \\ 0.3 & 0 \end{pmatrix}$. With a small

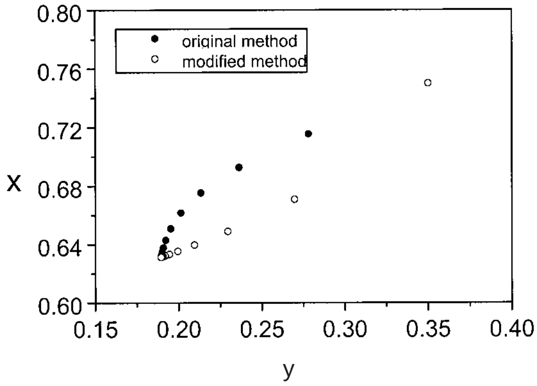

Fig. 1. The stabilization of the unstable fixed point $(0.63135, 0.18941)$ using both the original method and the modified one. The two orbits starting from the same initial point $(0.75, 0.35)$ converge to the unstable fixed point very quickly.

perturbation of the parameter (*i.e.* $\epsilon \neq 0$), the map has a fixed point in the neighborhood of (x_*^0, y_*^0), which is denoted by (x_*, y_*). We can obtain that

$$\mathbf{M} = \begin{pmatrix} \frac{\partial x_*}{\partial a} & \frac{\partial x_*}{\partial b} \\ \frac{\partial y_*}{\partial a} & \frac{\partial x_*}{\partial b} \end{pmatrix}_{a=a_0, b=b_0}$$

$$= \begin{pmatrix} \frac{2a_0 \Delta^{-1} + 1 - b_0 - \Delta}{2a_0} & \frac{1 + (b_0 - 1)\Delta^{-1}}{2a_0} \\ \frac{b_0(2a_0 \Delta^{-1} + 1 - b_0 - \Delta)}{2a_0} & \frac{b_0(1 + (b_0 - 1)\Delta^{-1})}{2a_0} + x_*^0 \end{pmatrix},$$

where $\Delta = \sqrt{(1-b_0)^2 + 4a_0}$. Therefore the control signal should be that

$$\epsilon = \epsilon_n = \begin{pmatrix} \delta a_n \\ \delta b_n \end{pmatrix} = \mathbf{M}^{-1}(\mathbf{J} - \mathbf{I})^{-1}(\mathbf{J} - k\mathbf{I}) \begin{pmatrix} x_n - x_*^0 \\ y_n - y_*^0 \end{pmatrix}$$
$$= 3.97357 \begin{pmatrix} -1.43178 & 0.63135 \\ -0.11958 & 0.19930 \end{pmatrix} \begin{pmatrix} x_n - 0.63135 \\ y_n - 0.18941 \end{pmatrix}, \quad (16)$$

where $k = 0.5$. Numerical result is shown in Figure 1. The orbit starting from the point $(0.75, 0.35)$ converges to the unstable fixed point $(x_*^0, y_*^0) = (0.63135, 0.18941)$ very quickly under iterations of the map with $a = a_0 + \delta a_n, b = b_0 + \delta b_n$ given by equation (16).

In our modified method, we act the external control signal $\epsilon = (p, q)$ on the Henon map in the following form

$$x_{n+1} = 1 - a x_n^2 + y_n + p,$$
$$y_{n+1} = b x_n + q, \quad (17)$$

where $a = a_0 = 1.4, b = b_0 = 0.3$. When $\epsilon = (p, q) = (0, 0)$, an unstable fixed point is at $(x_*^0, y_*^0) = (0.63135, 0.18941)$. According to equation (9), it is easy to get that

$$\epsilon = \epsilon_n = \begin{pmatrix} p_n \\ q_n \end{pmatrix} = (k\mathbf{I} - \mathbf{J}) \begin{pmatrix} x_n - x_*^0 \\ y_n - y_*^0 \end{pmatrix}$$
$$= \begin{pmatrix} 2.26779 & -1 \\ -0.3 & 0.5 \end{pmatrix} \begin{pmatrix} x_n - 0.63135 \\ y_n - 0.18941 \end{pmatrix}, \quad (18)$$

where $k = 0.5$. Numerical result is also shown in Figure 1. The orbit starting from the point $(0.75, 0.35)$ converges to the unstable fixed point $(x_*^0, y_*^0) = (0.63135, 0.18941)$ very quickly under iterations of the map with (p_n, q_n) given by equation (18).

From the example, we can see that our modified method presents a more convenient approach for controlling chaos.

The work is fewer for controlling Hamiltonian chaos than dissipative chaos. Without loss of generality, we choose the standard map [12] as an application, which has become a paradigm for the study of properties of chaotic dynamics in Hamiltonian systems

$$J_{n+1} = J_n - \frac{\beta}{2\pi} \sin(2\pi\theta_n), \qquad (\text{mod } 1)$$
$$\theta_{n+1} = \theta_n + J_n - \frac{\beta}{2\pi} \sin(2\pi\theta_n) \quad (\text{mod } 1). \quad (19)$$

If we select the system parameter β to be the control parameter, we obtain $\mathbf{M} = \mathbf{0}$ for the unstable fixed point $\mathbf{x}_*^0 = (J_*^0, \theta_*^0) = (0, 1/2)$, in which the original straight-line stabilization method cannot be used. In this case, we use our technique and act the external control signal $\epsilon = (p, q)$ on the standard map in the following form

$$J_{n+1} = J_n - \frac{1}{2\pi} \sin(2\pi\theta_n) + p, \qquad (\text{mod } 1)$$
$$\theta_{n+1} = \theta_n + J_n - \frac{1}{2\pi} \sin(2\pi\theta_n) + q \quad (\text{mod } 1). \quad (20)$$

In this paper, we let $\beta = 1$. When $\epsilon = (p, q) = (0, 0)$, an unstable fixed point of the period-1 orbit is at $\mathbf{x}_*^0 = (J_*^0, \theta_*^0) = (0, 1/2)$. It is easy to get that $\mathbf{J} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Therefore the control signal should be that

$$\epsilon = \epsilon_n = \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} k-1 & -1 \\ -1 & k-2 \end{pmatrix} \begin{pmatrix} J_n \\ \theta_n - 1/2 \end{pmatrix}$$
$$= \begin{pmatrix} (k-1)J_n - (\theta_n - 1/2) \\ -J_n + (k-2)(\theta_n - 1/2) \end{pmatrix}.$$

For an unstable period-10 orbit generated by successive iteration of the initial fixed point $(J_n, \theta_n) = (0.73577, 0.52669)$, the Jacobian matrix $\mathbf{J} = \begin{pmatrix} -0.64960 & -0.63680 \\ 7.04563 & 5.38366 \end{pmatrix}$. According to equation (12), the control signal

$$\epsilon = \epsilon_{n-9} \sum_{m=1}^{\infty} \delta(n + 1 - 10m)$$
$$= \begin{pmatrix} k + 0.64960 & 0.63680 \\ -7.04563 & k - 5.38366 \end{pmatrix} \begin{pmatrix} J_{n-9} - 0.73577 \\ \theta_{n-9} - 0.52669 \end{pmatrix}$$
$$\times \sum_{m=1}^{\infty} \delta(n + 1 - 10m).$$

Figures 2a and 2b show the stabilization of the unstable period-1 and period-10 orbits when $k = 1/2$ for the

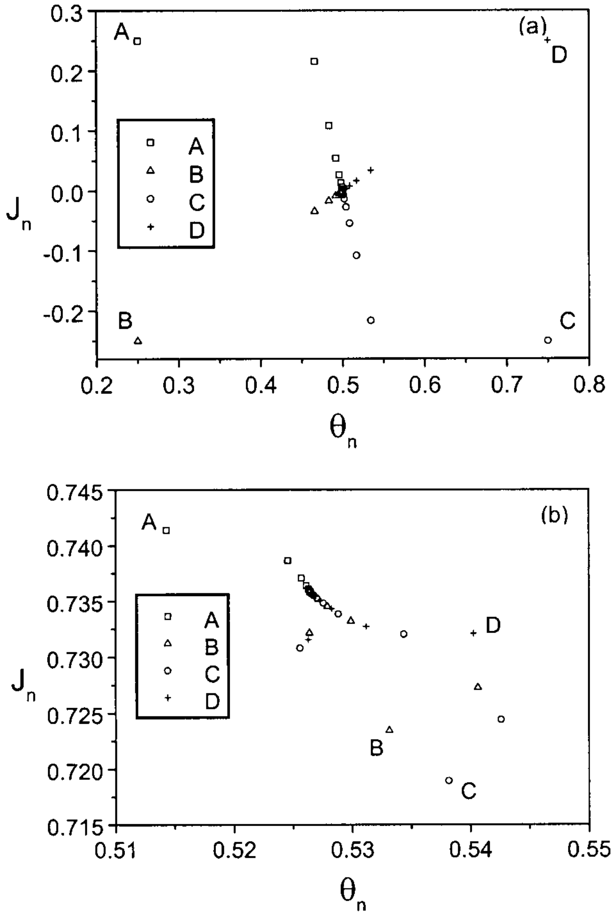

Fig. 2. The stabilization of the unstable period-1 and period-10 orbits when $k = 1/2$ for the standard map. Four orbits starting from points A, B, C, and D converge to the unstable fixed points $(0, 1/2)$ in (a) and $(0.73577, 0.52669)$ in (b), respectively.

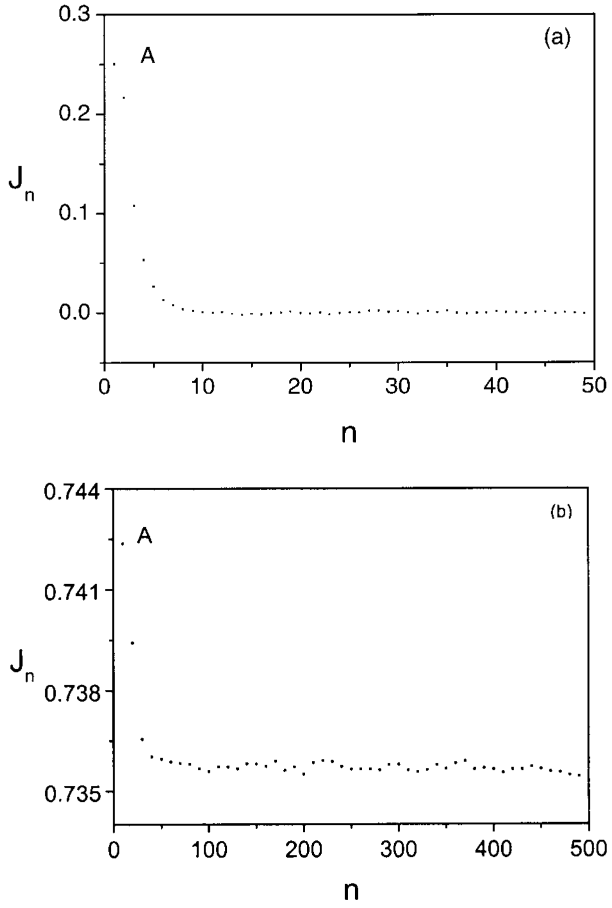

Fig. 3. The effect of noise on the controlled system corresponding to the period-1 and period-10 orbits, respectively. The intensities of noise are $\rho = 5.0 \times 10^{-4}$ in (a) and $\rho = 5.0 \times 10^{-5}$ in (b).

standard map, respectively. There are four orbits starting from points A, B, C, D, respectively. With the control signal adjustment, all the orbits converge to the unstable fixed point very quickly.

An important issue whether a control method is useful in experiments is its robustness against the application of external noise. In this section, we consider the Gaussian white noise generated by using the Box-Muller method [13], and introduce additive noise in the form

$$(x_i)'_n = (x_i)_n + \rho\xi_n, \qquad (i = 1, 2, \cdots, N) \tag{21}$$

where ρ denotes the intensity of external noise.

$$\langle \xi_n \rangle = 0, \qquad \langle \xi_n \xi_{n'} \rangle = \delta_{nn'}. \tag{22}$$

Figure 3 shows the effect of noise on the controlled system corresponding to the period-1 and period-10 orbits, respectively. The intensities of noise are $\rho = 5.0 \times 10^{-4}$ in (a) and $\rho = 5.0 \times 10^{-5}$ in (b). This shows that our method is robust against the presence of weak external noise. The numerical results also show that the effect of noise is more sensitive for higher periodic orbits than that for lower periodic orbits.

In conclusion, we have shown how an unstable periodic orbit in the neighborhood of an unstable fixed point can be stabilized. The unstable periodic orbit can be stabilized by adjusting the external control signal, rather than some available parameters of the system. We derive a simple analytical expression of the external control signal adjustment. Our technique avoids the case of $\mathbf{M} = \mathbf{0}$, in other words, our technique solves the problem that the unstable fixed point is independent of the system parameters. However, it is not always the case in equation (1) of reference [9]. This means that our modified method is more general than the original straight-line stabilization method in the sense. The new method is valid for controlling dissipative chaos, Hamiltonian chaos and hyperchaos. In particular, this method might be of application in those situations where finding an accessible system parameter may be a difficult task, namely, the case of certain chemical, biological, or spatially nonhomogeneous systems.

The method presents a possible solution to the problem of long chaotic transient which is very important in controlling Hamiltonian chaos. We also show that the method is robust against the presence of weak external noise, which is important for practical applications. The method can be generated to any high-dimensional system, including both discrete and continuous systems.

The work is supported by the Special Funds for Major State Basic Research Projects, the National Natural Science Foundation of China, and the Science Foundation of China Academy of Engineering Physics.

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